ON QUASI-MONOTONOUS GRAPHS

Mekkia KOUIDER

LRI, UMR 8623, University Paris-Sud and CNRS
F-91405 Orsay cedex, France

Abstract

A *dominating coloring* by k colors is a proper k coloring where every color i has a representative vertex x_i adjacent to at least one vertex in each of the other classes. The *b-chromatic number*, $b(G)$, of a graph G is the largest integer k such that G admits a dominating coloring by k colors.

A graph $G = (V, E)$ is said *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph H_1 of G and every subgraph H_2 of H_1.

Here we say that a graph G is *quasi b-monotonous* , or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$.

We show study the quasi-monotonicity of several classes. We show in particular that chordal graphs are not quasi-monotonous in general,whereas chordal graphs with large b-chromatic number, and $(P, coP, chair, cochair)$-free graphs are quasi-monotonous; (P_5, coP_5, P)-free graphs are monotonous.Finally we give new bounds for the b-chromatic number of any vertex deleted subgraph of a chordal graph.

Key words:

1 Introduction

All graphs considered here are simple and undirected. We denote by P_n (respectively C_n) an elementary path (resp. an elementary cycle) with n vertices. Let G be a graph with a proper coloring. For any two disjoints subsets A and B, let $E(A, B)$ be the set of edges of G with one extremity in A and the other in B. Let u_i be any vertex u of color i. Let us denote by $\mathcal{C}_i$ the class of color i. If y is a vertex of the graph G, let $N_i(y)$ be the set of neighbours of y of color i; while for any integer p non zero, $N^p(y)$ is the set of vertices at distance exactly p from y.

Email address: mekkia.kouider@lri.fr (Mekkia KOUIDER).

In a proper coloring, a vertex x_i of color i is said a *dominating vertex* if x_i adjacent to at least one vertex in each of the other classes. The vertex x_i is also called a dominant. The color i is said *dominating* if there exists at least a vertex of color i which is dominating. A *dominating coloring* by k colors is a proper k coloring where every color i has a at least a dominating vertex.

The *b-chromatic number*, $b(G)$, of a graph G is the largest integer k such that G admits a dominating coloring by k colors. A dominating coloring with $b(G)$ colors will be called a b-coloring.

This parameter was defined by Irving and Manlove [6]. They proved that determining $b(G)$ for an arbitrary graph G is an NP-complete problem.

For a given graph G, it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G)+1$. If we are limited to regular graphs, Kratochvil et al. proved in [7] that for a d-regular graph G with at least d^4 vertices, $b(G) = d + 1$. Kouider and El Sahili ([4]) proved that for every regular graph of girth 5 and no induced cycle C_6 the same equality holds.

Let v be any vertex of a graph G. It is known that $\chi(G - v) \leq \chi(G)$. The function χ is said monotonous. This is not the case for the b chromatic number.

A graph $G = (V, E)$ is called *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph H_1 of G and every subgraph H_2 of H_1. This was a definition of Bonomo et al.

Here we say that a graph G is *quasi b-monotonous* , or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$.

A *chordal* graph is a graph where every cycle of length at least 4 has at least one chord. A *quasi-line* graph is a graph where the neighborhood of every vertex has a partition into at most 2 cliques. A *P_4-sparse graph* is a graph where every 5-vertex subset contains at most one induced P_4.

It was shown by Bonomo and al.([2]) that P_4-sparse graphs are b monotonous. On the other hand, we showed in [5] that every graph of girth at least 5 is b-monotonous.

Clique-width A parameter used in complexity of graphs is the clique-width. Many problems of optimisation which are NP Hard can be solved efficiently on graphs with bounded clique-width. Some of the classes with bounded clique-width are defined by forbidden subgraphs on 5 vertices, among P_5,coP_5, P , chair, cochair, coP (see fig. 1)([3]).

The class of $(P_5, coP_5, cochair)$-free graphs was defined, by Giakoumakis and

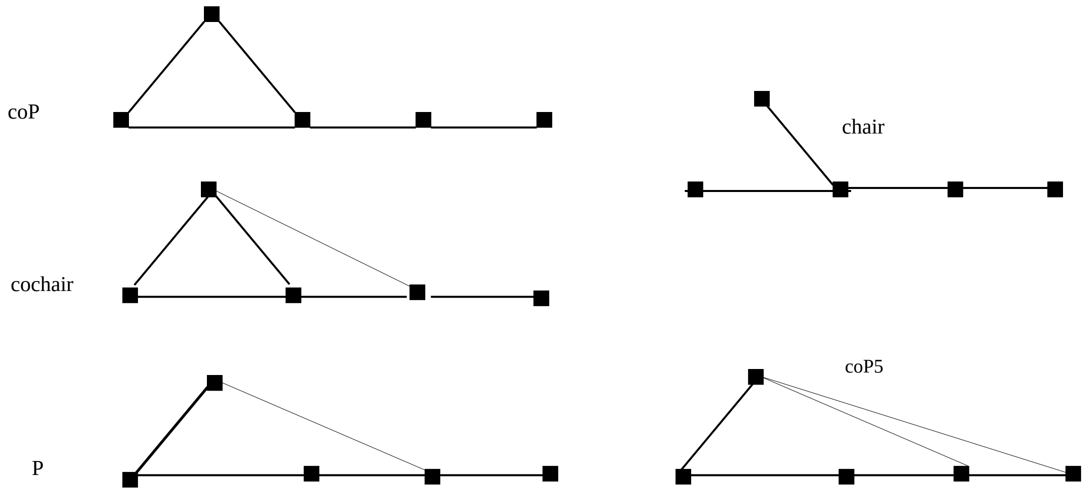

Fig. 1. Extensions of P4

Fouquet,as the class of semi-P_4-sparse graphs, a superclass of P_4-sparse graphs.The class $(P_5, coP, chair, cochair)$-free deserves the name of semi P_4-sparse.

For general graphs S.F.Raj and R.Balakrishnan proved that

Theorem 1 *[1] For every connected graph of order $n \geq 5$, and for every vertex $v \in V(G)$,*

$b(G) - (\lceil n/2 \rceil - 2) \leq b(G - v) \leq b(G) + (\lfloor n/2 \rfloor - 2)$

The upper bound is attained.

2 Quasi-monotonous graphs

Our main results are the following.

Theorem 2 *Let $G = (V, E)$ be a graph.*

1) If each vertex is contained in at most two cycles of length 4. Then G is quasi-b-monotonous.

2) If G is $(P, coP, Chair, Cochair)$-free, G is quasi-b-monotonous.

Corollary 1 *[5] Every graph of girth at least 5 is quasi-b-monotonous.*

Theorem 3 *Let $G = (V, E)$ be a graph.*

1) If G is $(P_5, P, Cochair)$-free, G is b-monotonous.

2) If G is (P_5, coP_5, P)-free, G is b-monotonous.

Theorem 4 *Let $G = (V, E)$ be quasi-line-graph. Then for each vertex x, $b(G - x) \leq b(G) + 2$*

Theorem 5 *Let $G = (V, E)$ be a chordal graph of clique-number ω.*

Then, for each vertex x of G,of degree $d(x)$,

$$b(G - x) \leq b(G) + 1 \ + \ \frac{d(x) - 1}{b(G - x)} \quad (1)$$

$$b(G - x) \leq b(G) + 1 + \ \frac{\omega - 1}{b(G - x) - \omega} \quad (2).$$

$$b(G - x) \leq b(G) + 1 + \frac{(\omega - 1)^{3/4}}{(b(G) - \omega)^{1/2}} \quad (3)$$

From the preceding theorem, we deduce

Corollary 2 *Let $G = (V, E)$ be a chordal graph of clique-number ω and b-chromatic number $b(G)$. Then, for each vertex x,*

$$b(G - x) \leq b(G) + 1 + \sqrt{d(x) - 1}$$

$$b(G - x) \leq b(G) + 1 + \sqrt{\omega - 1}.$$

Corollary 3 *Let $G = (V, E)$ be a chordal graph of clique-number ω and b-chromatic number b such that $b \geq 2\omega - 3$.*

Then G is quasi b-monotonous.

There exist chordal graphs and quasi-line graphs not b-monotonous.

Examples 1) Let $k \geq 3$ be an integer .

Let ω be an even integer and $2k$ be a divisor of ω, furthermore we suppose $\omega \geq 4.k^2$. We give an example of a chordal graph G_1 with dominating number $b = \omega + \dfrac{\omega}{2k} - k + 1$ and not quasi-b-monotonous. The gap $b(G_1 - x) - b(G_1)$ in this example is of the order of $\sqrt{\omega}$.

Consider the following graph H composed by 4 vertex-disjoint cliques A_1, A_2, A_3, A_4 such the order of A_1 (resp. of A_4) is $\omega - \omega/2k$. Furthermore, $A_1 \cup A_2$ is a clique of order ω as well as $A_4 \cup A_3$, and, $A_3 \cup A_2$ is a clique of order ω/k (see fig.2).Let us call H' the graph $H - A_4$.

The graph G_1 is a graph composed by k disjoint copies of H and a copy of H', and, with a vertex x, external to the copies of H and H', joined to every vertex of any copy of A_1 and to every vertex of H'.

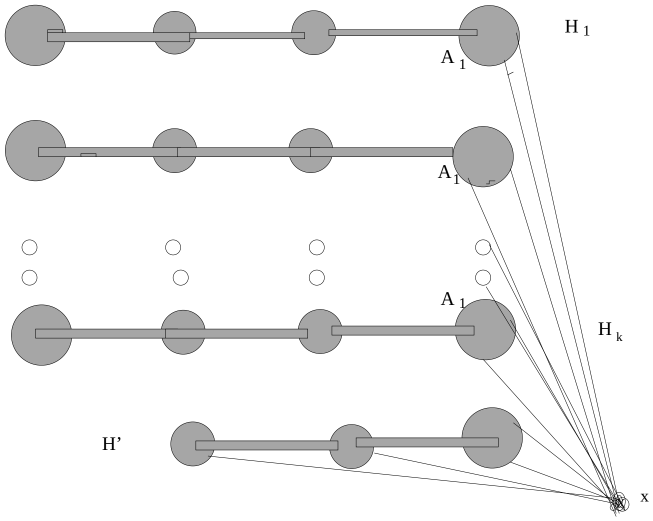

Fig. 2. Chordal graph not quasi-monotonous

We observe that

$$d(v) \leq \omega \ + 1$$

for every vertex v of H';

$$d(x) = k.\omega \ + \ \omega/2 + \omega/2k;$$

and,

$$d(u) = \omega \ + \ \omega/2k \ - 1$$

for every vertex u of a copy of A_2 in G_1 or every vertex of a copy of A_3 in $G_1 \ - \ H'$. The other vertices have degree at most ω. The number of vertices of degree at least $\omega + \omega/2k \ - k - 1$ is

$$\omega + \omega/2k \ + 1 \qquad (a)$$

The graph $G_1 - x$ has a b-chromatic number equal to $\omega + \omega/2k$, the set of dominating vertices is the set of vertices of maximum degree in$G_1 - x$. We show that the graph G_1 has a b-chromatic number equal to $\omega + \omega/2k \ - k + 1$.

Indeed, suppose $b(G_1) \geq \omega + \omega/2k \ - k$, so $b(G_1) \geq \omega + 3$. Given a b coloring of G, at least $\omega/2k \ - k - 1$ colors have a dominating vertex in H'; then x is neighbour of $b-1$ colors, and is dominating. Every dominating vertex outside H' must be neighbour of the color $c(x)$. And as $\omega/2k \ - (k+1) > 0$, each copy of A_2 and each copy of A_3 outside H' must contains a dominating vertex; as x is neighbour of each copy of A_1, this implies that each copy of $A_2 \cup A_3$ outside

H' must contain a vertex of color $c(x)$. Then by (a), $b(G_1) \leq \omega+\omega/2k \; -k+1$. One can verify easily the equality $b(G_1) \leq \omega+\omega/2k \;\; -k+1$. So $b(G_1-x) = b(G_1) \; + \; k-1$.

In that example, if $\omega = 4k^2$, then

$$b(G_1 - x) = b(G_1) - 1 + \; \frac{\sqrt{w}}{2}.$$

2) With the notations of the precedent example, we consider a graph G_2 composed by two vertex disjoint copies of H, and an external vertex x joined to each copy of A_1. We have $|A_1| = |A_4| = 2\omega/3$ and $|A_2| = |A_3| = \omega/3$.

Then G_2 is a quasi-line graph and $b(G_2 - x) = b(G_2) + 1$.

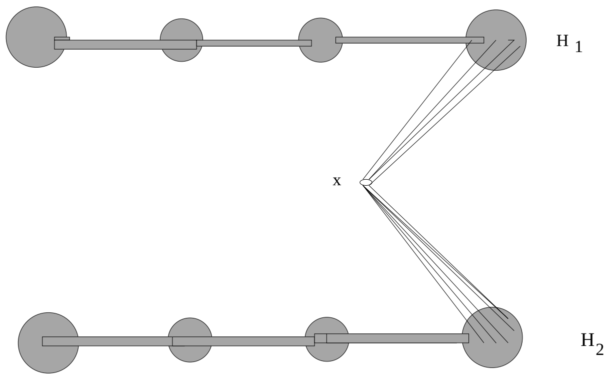

Fig. 3. Quasi-line graph not monotonous

Remark: There exist chordal graphs with b-chromatic number at least 2ω. Let $\omega \geq 2$, $p = 5(\omega-1)$. Let $S(u)$ be the graph composed by 5 cliques sharing exactly one vertex u. Consider the graph H composed by an elementary path $P(u_1, ..., u_p)$ and a family of p graphs $S(u_i)$ for $1 \leq i \leq p$. It is easy to see that H is chordal and $b(H) \geq p \geq 4\omega$.

3 Proofs

Remarks For each of our results the following remarks are valid. The proofs are by contradiction. Let x be a fixed vertex of G. Let $H = G - x$. We may suppose first that $q = b(H) \geq \omega \; + \; 2$ otherwise we have $b(H) \leq b(G) + 1$. Suppose the b-coloring of H is not extendable to G. Then, necessarily,

R1) All the colors appear in $N(x)$.

R2) x is not neighbour to q dominating vertices s of different colors, so there exists an integer r such that for every color $i \geq r+1$, $N_i(x)$ contains no dominanting vertex.

R3)Let us consider r minimum. For each $i \geq r+1$, $N(N_i(x)) \cap N^2(x)$ contains all the dominating vertices of some color j and no one of these dominating vertices has a neighbour of color i outside $N_i(x)$; otherwise for each $y \in N_i(x)$, we change the color i of the vertex y into a color missing in the neighborhood of y and we give color i to the vertex x and we obtain a b-coloring of G with q colors. Let us call I the set colors $i, i \geq r+1$.

Note that, as the coloring is proper and $j > r$,we get $q - r \geq 2$. From now we say that a set $A \in N_i(x)$ *covers* a color j if $N(A)) \cap N^2(x)$ contains all the dominating vertices of some color j and no one of these dominating vertices has a neighbour of color i outside A. Note that $j \in I$ necessarily.

Now we give first the proof of theorem2.

Proof of Theorem 2

Proof of the first part of the theorem2

Case 1:*There exists $i \geq r+1$ such that $N_i(x)$ covers at most one color, say j*

For each $y \in N_i(x)$, we change the color i of y into a color missing in the neighborhood of y.

We x by i. Either there remains a dominating vertex of the color j, we keep color j. Or the color j has no dominating vertex.For each vertex of color j we give a color missing in its neighborhood. In any case, we get a b coloring of G with at least $q - 1$ colors..

Case 2:For each $i \geq r+1$, $N_i(x)$ covers at least 2 different colors.

Let b' the number of colors with all dominating vertices contained in $N^2(x)$ and neighbours of $\cup_{i \geq r+1} N_i(x)$. As the coloring is proper, and $q - r \geq 2$, then $b' \geq 3$ by definition of case 2; so $b - r \geq b' \geq 3$. Let S be a system of b' dominating vertices of the b' different colors.

There exist at least $2(q - r)$ edges between $\cup_{i \geq r+1} N_i(x)$ and the set S, by definition of case 2.Then either there exists a vertex s_0 which sends at least 3 edges to $\cup_{i \geq r+1} N_i(x)$, or each vertex of S sends exactly 2 edges to $\cup_{i \geq r+1} N_i(x)$. As $|S| \geq 3$, there are at least 3 cycles of length 4 containing x. A contradiction to the hypothesis on G.

This ends the proof of part1 of theorem 2.

Proof of part 2 of Theorem 2

We use the proof of the first part. And we may suppose we are in case 2. For each color $j, j \geq r+1$, $N_j(x)$ covers at least 2 colors. Let $i, i \geq r+1$ be a color such that a dominating vertex x_i is in $N^2(x)$, neighbour of $u_t \in N(x)$. Let us note that, by definition of the covering,x_i is independent of the dominants of each color covered by $N_i(x)$.

There exists a vertex $y_i \in N_i$ covering a color k, otherwise, there are y_i and y'_i in $N_i(x)$, x_k and x'_k dominants of the same color such that the path $Q = \{x_k, y_i, x, y'_i, x'_k\}$ is induced and forms with $u_k \in N_k(x)$ a chair,a coP or a cochair; and we get a contradiction with the hypothesis.

Furthermore,there is a second color, say s, covered by $N_i(x)$; either s is covered by y_i, then as there is no chair the dominants x_k and x_s are adjacent; or$|N_i(x)| \geq 2$, and there exists $y'_i \in N_i(x)$ neighbour of a dominant $x_s, s \neq k$ independent of y_i; x_k is independent of y'_i otherwise x_i, y'_i, x, y_i, x_k should be a P.

Case α: x_k, u_t adjacent

The set u_t, x, y_i, x_i, x_k forms P if y_i, u_t are independent, or a cochair if y_i, u_t are adjacent.

Case β: x_k, u_t independent

Case α being excluded, we may suppose that x_s, u_t are independent too. (a) If y_i, u_t are independent,

we have a chair or coP composed by u_t, y_i, x_s, x_k, or if y'_i exists, a chair or a cochair composed by u_t, x, y_i, y'_i, x_k.

(b) If y_i, u_t are adjacent,

Either y_i is adjacent to x_k, x_s, we have coP composed by x_i, u_t, y_i, x_s, x_k.

Or, y_i is adjacent to x_k and not to x_s, and y'_i is adjacent to x_s and not to x_k otherwise we are in some precedent case. Then y_i, u_t, x, y'_i, x_s forms either a coP (if $[y'_i, u_t] \notin E(G)$) or a cochair (if $[y'_i, u_t] \in E(G)$). As the subgraphs P,coP,chair,cochair are excluded, in any subcase, there is a contradiction. Case2 cannot occur.We get part2 of the theorem. $\square$

Proof of theorem 3

1) Each class considered here is hereditary. It is sufficient to show by contradiction that for any graph G of the class and any vertex x of G a dominating coloring of $G - x$ extends to G, so $b(G - x) \leq b(G)$.

We keep the notations of the proof of the second part of Theorem 2. There exists at least a vertex y_i such that $N(y) \cap N^2(x)$ contains all the dominants of at least a color k, otherwise we know that there is at least a color k with all dominants in $N(N_i(x)) \cap N^2(x)$ and we get a P_5 composed by $xy_iy_i'x_kx_k'$ where y_i, y_i' are in $N_i(x)$ and x_k, x_k' are 2 dominants of color k, a contradiction with the hypothesis on G . By remark R3, no dominating vertex of the color k has a neighbour of color i outside $N_i(x)$.

1) *Case α: y_i, u_t independent*

We have a P_5 or P composed by x_k, y_i, x, u_t, x_i.

Case β: y_i, u_t adjacent

As u_t is not dominant, there is a color s and a vertex $z_s \in N(x)$ not neighbour of u_t. As x_i is dominant,there exists a vertex v_s neighbour of x_i. We may suppose x_k, u_t independent otherwise we have a cochair x_k, y_i, u_t, x_i.

subcase:z_s neighbour of x_i:

(a) Either x_k is not neighbour of z_s, then x_k, y_i, u_t, x_i, z_s gives either P_5 or P;

(b) Or, x, u_k, z_s, x_i, x_k form P.

subcase:z_s, x_i independent :

As u_t and v_s are independent, then x, u_t, v_s, x_i and z_s form either P_5 or P. As the graph is $(P_5, P, cochair)$ free, there is a contradiction in any case. So there exists a color j such that $N(N_j(x)) \cap N^2(x)$ contains no dominant. 2) The proof is similar to that of part1 . The only difference is that in the subcase where x_k and y_i are adjacent to u_t, we have a coP_5 composed by y_i, u_t, x_i, v_s, x □

Proof of theorem4

Let i be a color in I. As G is a quasi-line graph,there are at most two neighbours u_i and u_i' of x which are of color i; $N^2(x) \cap N(u_i)$ is a clique, the same holds for $N^2(x) \cap N(u_i')$ if u_i' exists. By remarque R3, there exists at least a color $j \in I$ with all dominating vertices in $N^2(x) \cap (N(u_i) \cup N(u_i'))$.

We do the following operation:

1) We change the color of u_i into a missing color $p(u_i)$ and that of u_i' into a missing color $p(u_i')$. We color x by i.

2)We choose one color, say s, which is no more dominating

a) either the initial dominating vertex w_s was unique and in $N^2(x) \cap N(u)$,where $u \in \{u_i, u'_i\}$, we recolor w_s by the color i.

Furthermore, if there is a color s' which the initial dominating vertex $w_{s'}$ was unique and in $N^2(x) \cap N(v)$ where $v \in \{u_i, u'_i\}, v \neq u$, we choose one such color and we do the same operation as precedently on $w_{s'}$.

b) case (a) being excluded, the color s had exactly 2 initial dominating vertices and they were in $N^2(x) \cap (N(u_i) \cup (N(u'_i))$, we recolor them by i.

3) For each color, s or s', we recolor each vertex of the corresponding class by a missing color.

Thus, after steps(1) and (2) of this operation, we get a proper coloring; the vertex x is so a dominant vertex of the color i;furthermore, each neighbour of u_i (resp. of u'_i) which is not of color i is neighbour of a vertex of color i. At most 2 colors of the initial coloring are not used, namely s and s'. After step (3), we have a dominating coloring of G with at least $q-2$ colors.

This finishes the proof of theorem4 $\square$

Proof of theorem 5 We suppose we have a dominating coloring of $H = G - x$. We use notations and remarks R1, R2,R3 given upper, in the proof of Theorem2. For each color i in I, we choose a dominating vertex w_i. Let W be the set of these dominating vertices. Let $t, t \geq r+1$ be a fixed color. Let y in $N_t(x)$, and let $p(y) \neq t$ a color not neighbour of y. If no ambiguity, we shall write simply p instead of $p(y)$. We call $\mathcal{C}_{t,p}(y)$ the set of vertices joined to y by a path with vertices in $\mathcal{C}_t \cup \mathcal{C}_p \cup W$ such that no 2 vertices of W are consecutive.

Let us first describe the operation $\mathcal{O}$ on $N(x)$.

We fix a color t.

O_1) Process $O_1(y)$: If $y \in N_t(x)$ is not neighbour of $N_p(x)$ for some p, we choose such a color and we color y by p . We exchange the two colors t and p in the component $\mathcal{C}_{t,p}(y)$.
We do this operation successively for each vertex y of $N_t(x)$.
O_2) Finally, we give the color t to x.

We remark that:

As G is a chordal graph, two vertices of $N_t(x)$ have no common neighbour outside $x \cup N(x)$; furthermore, there is no path $P(y, N_t(x))$ with internal vertices in $\mathcal{C}_{(t,p)}(y)$, and, no path $P(y, N_p(x))$ in $\mathcal{C}_{(t,p)}(y)$;so $\mathcal{C}_{(t,p)}(y)$ does not meet $x \cup N(x)$ outside y.

The operation $\mathcal{O}$ is possible for every color t with no dominating vertex in $N(x)$.

Lemma 1 *Let $t \geq r+1$ fixed.*

1)After an application of operation $O_1(y)$, we obtain a proper coloring of G; at most one element of W, w_t or w_p, is no more dominant. If the color $p(y)$ is no more dominating, then $w_p \in \mathcal{C}_{t,p(y)}(y)$

Furthermore, if $y, y' \in N_t(x)$, $y \neq y'$, then $\mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y') = \emptyset$.

2)After operation $\mathcal{O}$, the vertex x is a dominating vertex of color t. The colors which have no dominating vertex are among the chosen missing colors.

Proof of Lemma 1

1)If for some $j \neq p(y), j \neq t$, $w_j \in \mathcal{C}_{t,p(y)}(y)$, then necessarily its neighbours of colors $p(y)$ and t are also in $\mathcal{C}_{t,p(y)}(y)$; then by operation $O_1(y)$, there is a permutation of the colors $p(y)$ and t in $\mathcal{C}_{t,p(y)}(y)$;so the vertex w_j remains dominating of color j. We remark that if $w_j \notin \mathcal{C}_{t,p(y)}(y)$, then its neighbours of colors $p(y)$ and t are not in $\mathcal{C}_{t,p(y)}(y)$; their colors are not changed after operation $O_1(y)$, the same conclusion holds for w_j; so vertex w_j remains dominating of color j.

If $w_t \in \mathcal{C}_{t,p(y)}(y)$, then after operation $O_1(y)$, the vertex w_t becomes dominating of color p. Analogously, if $w_p \in \mathcal{C}_{t,p(y)}(y)$, after operation $O_1(y)$, w_p becomes a dominating vertex of color t.

Suppose $u \in \mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y')$. Then there exists a path from u to y and another one from u to y', so there is a cycle C containing the induced path $[y, x, y']$ and no vertex of $C - \{y, y'\}$ is neighbour of x. We may suppose C is a shortest cycle with the latest properties . C is of length at least 4 and has no chord. As G is chordal we get a contradiction.

2) After operation $\mathcal{O}$, the vertex x is neighbour of every color except t; so x is dominating vertex of color t. As $\mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y') = \emptyset$, any vertex is recolored at most one time and by 1) of the Lemma, the possible non dominating colors are in the set $(p(y))_{y \in N_t(x)}$ •

Let R_t be the set of vertices of $\mathcal{C}_t$ such that for each vertex y of R_t for any missing color $p(y)$, after the operation $\mathcal{O}$, the color $p(y)$ has no dominating vertex. Furthermore, if $\mathcal{C}_t = \mathcal{C}$, let us denote R_t simply by R.

Lemma 2 *Let G be a chordal graph. Then, for each $t \in I$,*

1) $b(G-x) \leq b(G) + |R_t|$

2)For any vertices y, y' of R_t, then $p(y) \neq p(y')$. And, if $d^-(y)$ is the number of colors which do not appear in $N(x) \cap N(y)$, we have

$$\sum_{y \in R_t} d^-(y) \leq (q-1)$$

.

Proof of Lemma 2

1) The first part is a consequence of (2) of Lemma1.

2)If y and y' are elements of R_t, there exists a path $P(y, w_{p(y)})$ in $\mathcal{C}_{(t,p(y))}$, and a path $P(y, w_{p(y')})$ $\mathcal{C}_{(t,p(y'))}$ in by Lemma 1. As $\mathcal{C}_{(t,p(y))} \cap \mathcal{C}_{(t,p(y'))} = \emptyset$, then by Lemma1, they have no common missing color j, so we get the inequality $\bullet$

End of the proof of theorem 5

We consider a b-dominating coloring of $G-x$ by $q = b(G-x)$ colors. After operation $O_1(v)$ applied on a vertex v, the color $p(v)$ remains dominating if $N_t(x) = v$ was missing only one color in G and v is a reprentative of $p(v)$.

In $N(x)$, let $\mathcal{C}$ be a class of colors such that by the operation O the minimum number of colors are no more dominating.

In a chordal graph, if S_1 and S_2 are two vertex disjoint stable sets, the induced vertex graph of vertex- set $S_1 \cup S_2$ is a forest. Then given a coloring of $N(x) \setminus \mathcal{C}$ by a minimum number of colors, we have a partition $X_1, ..., X_s$ of $N(x) \setminus \mathcal{C}$. Then, as there is no induced C_4, the set of edges $E(R, X_i)$ is a matching between R and X_i. So

$$e(R, X_i\ ) \leq min(|R|,\ |X_i|)$$

for each $i \leq s$.

As $N(x) - \mathcal{C}$ is chordal of clique-number at most $(\omega - 1)$, we get by summing

$$e(R, N(x) - \mathcal{C}) \leq min((\omega-1).|R|, (d(x) - |R|)) \qquad (5)$$

Each vertex y of $\mathcal{C}$ is not neighbour in $N(x)$ of at least $(q-1) - d_{N(x)}(y))$ colors. So

$$(q-1)|R| - \ e(R, N(x) - \mathcal{C}) \leq \sum_{y \in R} d^-(y).$$

On the other hand, as G is chordal,by Lemma 2,

$$\sum_{y\in R} d^-(y) \leq (q-1),$$

we get, using inequality (5), that

$$(q-1)|R| - (d(x) - |R|] \leq q-1$$
$$and, (q-1)|R| - (\omega-1)|R| \leq q-1$$

So $|R| \leq 1 \ + \ min(\frac{d(x)-1}{q}, \frac{(\omega-1)}{(q-\omega)})$

As by Lemma2, $b(G-x) \leq b(G) + |R|$, we get the inequalities 1 and 2 of the theorem.

Let us set $\theta = b(G-x) - b(G) - 1$, and $a = q - \omega$. From inequality (2), we get

$$\theta^2 + a.\theta - (\omega - 1) \leq 0.$$

It follows that

$$\theta \leq \frac{(\omega-1)}{\sqrt{a^2/4 + (\omega-1)} + a/2} \qquad (6)$$

As $\sqrt{a^2/4 + (\omega-1)} \geq \sqrt{a}.(\omega-1)^{1/4}$, we get the third inequality of the theorem $\square$

Proof of corollary 2 From inequality 1 of the last theorem, we have $(\theta+1)^2 \leq d(x) - 1$. Whereas inequality (6) of the last proof gives

$$\theta \leq \frac{(\omega-1)}{\sqrt{a^2/4 + (\omega-1)} + a/2} \qquad (6)$$

Proof of corollary 3 Let us set $\theta = b(G-x) - b(G)$. The inequality (2) of the last theorem gives

$$(\theta-1)(b(G) - \omega + 3) \leq \omega - 1$$

Suppose $\theta \geq 2$. Then we get $b(G) \leq 2\omega - 4$. The corollary 2 follows •

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